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presented:

# Edge deletion to tree-like graph classes


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## Preliminaries

- Simple undirected graphs unless stated otherwise
- Let $V' \subseteq V$, then $N(V')$ is a neighborhood of $V'$ and $G[V']$ is a graph induced by $V'$
- A vertex is *universal* if it is adjacent to all the vertices in a graph
- For any family $\mathcal{F}$ of graphs, we say that $G$ is *$\mathcal{F}$-free if* $G$ does not contain any graph $F \in \mathcal{F}$ as an induced subgraph; the graphs in $\mathcal{F}$ are called the *forbidden induced subgraphs*
- A graph property (or a graph class) $\Pi$ is *hereditary* if $\Pi$ is inherited by every induced subgraph of a graph
- Given a graph class $\Pi$, we say that a subgraph $H$ of $G$ is a *maximum spanning* $\Pi$ if $H$ belongs to $\Pi$ and the number of edges is maximum among all the $\Pi$-subgraphs of $G$
- Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, $V_1 \cap V_2 = \emptyset$, then $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$ is called the *union* and $G_1 \oplus G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup V_1 \times V_2)$ is called the *join*

### Graph classes

| | |
|---|---|
| *subcubic graph* | Every vertex has degree at most three |
| *cactus* | Is connected and every edge of the graph lies in at most one cycle |
| *star* | A complete bipartite graph $K_{1,n}$ for some $n \geq 0$ |
| *claw* | A star $K_{1,3}$ |
| *constellation* | A forest where each connected component is a star if each connected component of a graph G induces a cactus, we say that G is a forest of cacti |
| *even-hole-free graph* | Contains no induced cycles with an even number of vertices |
| *chordal graph* | Contains no induced cycles of length greater than 3 |
| *cograph* | Is $P_4$-free; any cograph can be constructed from $K_1$ by repeated application of union and join operations |
| *caterpillar* | A tree in which the removal of all the leaves results in a path (called the spine or central path) |
| *quasi-threshold graph* | Is $\{P_4, C_4\}$-free; any quasi-threshold graph can be constructed from $K1$ by repeated application of disjoint union or addition of universal vertex |
| *interval graph* | There is an interval on the real line for each vertex and there is an edge if and only if the two intervals intersect |
| *kth power of a graph* | A graph $G^k$ having the same vertex set as $G$ such that any two vertices are adjacent if and only if the distance between them is at most $k$ |
| *split graph* | A graph in which the vertices can be partitioned into a clique and an independent set |
| *outerplanar graph* | A graph that has a planar drawing for which all vertices belong to the outer face of the drawing |

### Graph problems

- The Partition to P3 (PIP3) problem asks whether there exists a partition of $V$ into vertex-disjoint $P_3$'s
- The Hamiltonian Path problem asks whether there is a hamiltonian path in the graph, that is, a path that visits each vertex of a graph exactly once
- The Dominating Set problem asks whether there is a subset $D \subseteq V$ of some size such that each vertex in $V \setminus D$ is adjacent to at least one vertex in $D$

# Problem of interest

| Π-DELETION | |
|---|---|
| Input: | Graph $G$; $k \in \mathbb{N}$ |
| Question: | For a fixed property Π (that usually represents membership to a graph class), decide whether it is possible to modify the graph $G$ by deleting at most $k$ edges to obtain a new graph that fulfills the property Π |

The more general version of the problem is called Π-EDITING and also allows adding edges to the graph. The version of the problem which allows deletion of vertices instead of edges in called the Π-VERTEX-DELETION problem.

In optimization versions of the problems, the goal is to find a minimum modification of the input graph.

# Theorems

**Theorem 1.** DELETION TO CACTUS is *NP-complete* even for bipartite graphs.

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**Theorem 2.** DELETION TO CACTUS can be computed in polynomial time when the input graph is chordal and there is a polynomial combinatorial algorithm for computing minimum cactus deletion on connected quasi-threshold graphs.

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**Theorem 3.** Let Π be a graph class such that all of the following conditions hold:

(i) Π contains $P_n$ for all $n$
(ii) Π is claw-free
(iii) Π is even-hole-free

Then Π-deletion is *NP-hard* even when the input is restricted to planar bipartite graphs of maximum degree 3, or to chordal bipartite graphs.

**Corollary.** The Π-deletion problem is *NP-hard* when Π is the class of proper interval graphs, claw-free chordal, power of paths, even if the input is a bipartite graph.

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**Theorem 4.** DELETION TO CONSTELLATION is *NP-complete* for general graphs.

**Remark.** DELETION TO CONSTELLATION remains hard for any graph classes for which DOMINATING SET is hard. In particular, the problem remains hard for planar bipartite graphs and split graphs.

**Corollary.** DELETION TO CONSTELLATION can be polynomially solved within cographs and outerplanar graphs.

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**Theorem 5.** DELETION TO CATERPILLAR is *NP-complete* for general graphs.

**Corollary.** DELETION TO CATERPILLAR remains hard for graphs classes that are closed for double subdivision of edges and for which HAMILTONIAN PATH is hard, which is the case for planar bipartite graphs.